Variable Separation Approach for the Sine-Gordon System

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Using the Bäcklund transformation and a variable separation approach with some arbitrary functions, three new types of solutions of the sine-Gordon system have been obtained. The excitations are localized as well as non-localized. E.g. solitoffs, dromions, multidromions, lumps, breathers, instantons, multivalued solitary waves, doubly periodic waves, etc., can be constructed on the basis of selecting the arbitrary functions properly. Also the interaction properties for all the possible localized excitations are of interest. In this paper, we discuss two elastic interactions. – PACS Ref: 05.45.Yv, 02.30.Jr, 02.30.Ik.

Key words: Solitons; Sine-Gordon System; Variable Separation Approach; Interaction.

1. Introduction

Completely integrable nonlinear partial differential equations, that is, evolution equations of soliton type, have been studied now for almost forty years. They appear, and are important and relevant, in many different branches of applied mathematics and physics. The introduction of the concept of dromion in 1990, which is nothing but a two-soliton solution consisting of two non-parallel ghost solitons, has triggered renewed interest in (2+1)-dimensional soliton systems [1 – 3]. Recently, a variable separation approach (VSA) has been introduced which allows to construct new localized excitations for a class of nonlinear models [4 – 16]. The key idea of this method is to change the model to a multi-linear form with an arbitrary seed solution and then extend Hirota's two-soliton solution to a general variable separation ansatz. Finally, by substituting the variable separation ansatz to the original model and selecting the seed solution properly, one finds some variable separated solutions.

Here, the VSA procedure is adapted to the (2+1)-dimensional sine-Gordon system. We shall briefly describe the derivation of this system. In 1991, Konopelchenko and Rogers [17] constructed a (2+1)-dimensional master soliton system via a reinterpretation and generalization of a class of infinitesimal Bäcklund transformations originally introduced in a gasdynamics context by Loewner [18]. A particular reduction leads to a symmetric integrable extension of the classical 2dim sine-Gordon system, namely

$$\begin{aligned}
&(\phi_\xi \sin^{-1}\varphi)_\xi - (\phi_\eta \sin^{-1}\varphi)_\eta \\
&\quad +(\phi_\eta \varphi_\xi - \phi_\xi \varphi_\eta)\sin^{-2}\varphi = 0,\\
&(\phi'_\xi \sin^{-1}\varphi)_\xi - (\phi'_\eta \sin^{-1}\varphi)_\eta \\
&\quad +(\phi'_\eta \varphi_\xi - \phi'_\xi \varphi_\eta)\sin^{-2}\varphi = 0,\\
&\varphi_t = \phi + \phi'.
\end{aligned} \tag{1}$$

Introducing the new independent variables

$$x = (\eta - \xi)/2, \quad y = (\eta + \xi)/2, \tag{2}$$

and new dependent variables u,v into (1) via the relations

$$v_{xt} = \frac{\phi'_y - \phi_y - \varphi_{yt}\cos\varphi}{2\sin\varphi}, \quad u = \varphi/2. \tag{3}$$

leads to a compact version of the 2dim sine-Gordon system, namely

$$u_{xyt} + u_y v_{xt} + u_x v_{yt} = 0, \tag{4}$$

$$v_{xy} - u_x u_y = 0. \tag{5}$$

Since the discovery of the 2dim sine-Gordon system a decade ago, it has been the subject of intense investigation and many types of soliton solutions have been studied by many authors [19 – 28]. For instance, Chow [24] obtained the doubly periodic wave solutions. Lou constructed the localized solutions via a generalized bilinear operator representation. Radha and Lakshmanan studied the Painlevé property for the 2dim sine-Gordon system and have constructed dromion solutions. It turns out that the system contains particular reductions to the PI, PIII and PV transcendents [27]. As a physical application it contains the important pumped Maxwell-Bloch system. However, because of its complex structure a complete understanding of the model has not yet been achieved. Thus there might be a rich variety of excitations which have not yet been revealed. In addition, different types of solutions can be obtained from different separations of variables. Motivated by these reasons, we revisited the 2dim sine-Gordon system again by the Hirota-operator based approach to obtain three new types of variable separated solutions. We stress that the arbitrary functions p, p_1, p_2 introduced below are only functions of $\{x,t\}$ and the arbitrary functions q, q_1, q_2 are only functions of $\{y,t\}$.

2. Variable Separated Solution for the 2dim Sine-Gordon System

To use the VSA, we first perform a Bäcklund transformation

$$u = -2\tan^{-1}\left(\frac{f}{g}\right) + u_0, \quad v = \ln(f^2+g^2) + v_0, \quad (6)$$

where $\{u_0, v_0\}$ is an arbitrarily chosen known seed solution of the 2dim sine-Gordon system. We consider the special case

$$u_0 = 0, \quad v_0 = v_1(x,t) + v_2(y,t). \quad (7)$$

Substituting (6) with (7) into (4) and (5) gives rise to the bilinear forms

$$(D_xD_yD_t + v_{1xt}D_y + v_{2yt}D_x) f\cdot g = 0, \quad (8)$$

$$D_xD_y(f^2+g^2) = 0, \quad (9)$$

where the Hirota's bilinear operators D_x, D_y, D_t are defined by

$$\begin{aligned} &D_x^n D_y^m D_t^k f\cdot g \\ &\equiv \partial_{\varepsilon_1}^n \partial_{\varepsilon_2}^m \partial_{\varepsilon_3}^k f(x+\varepsilon_1, y+\varepsilon_2, t+\varepsilon_3) \\ &\quad \cdot g(x-\varepsilon_1, y-\varepsilon_2, t-\varepsilon_3)|_{\varepsilon_1=0,\varepsilon_2=0,\varepsilon_3=0}. \end{aligned} \quad (10)$$

Second, we will select a special ansatz for f and g. Different types of solutions can be obtained from different separation variables. We can derive a rich variety of solutions by selecting different expressions of f and g provided these choices solve the given equations. In order to clarify this, we will focus on listing three choices of seed solutions, which we describe below.

Assuming

$$\begin{aligned} f &= a_1p_1(x,t)q_1(y,t) + a_2p_2(x,t)q_2(y,t), \\ g &= b_1p_1(x,t)q_1(y,t) + b_2p_2(x,t)q_2(y,t), \end{aligned} \quad (11)$$

where $a_i, b_i (i=1,2)$ are taken as constants. Evidently the variables x and y have been separated by this ansatz. Substituting (10) into (9), we obtain the simple relation

$$a_1a_2 + b_1b_2 = 0. \quad (12)$$

Due to (11) and (12), Eq. (8) becomes

$$\begin{aligned} &2(-b_2a_1+a_2b_1)[(a_1^2+b_1^2)p_1^2q_1^2 + (a_2^2+b_2^2)p_2^2q_2^2] \\ &\cdot[(-p_2p_{1tx} - p_2v_{1tx}p_1 + p_{1x}p_{2t} + p_{2x}p_{1t} - p_{2tx}p_1) \\ &\cdot(-q_{1y}q_2 + q_{2y}q_1) + (p_{2x}p_1 - p_{1x}p_2) \\ &\cdot(-q_1v_{2ty}q_2 - q_{1ty}q_2 + q_{2t}q_{1y} + q_{2y}q_{1t} - q_1q_{2ty})] = 0. \end{aligned} \quad (13)$$

Apparently, (13) is correct only under the following choices

$$\begin{aligned} v_{1tx} = &-[p_2p_{1tx} - p_{1x}p_{2t} - p_{2x}p_{1t} + p_{2tx}p_1 \\ &+ r(t)(p_{2x}p_1 - p_{1x}p_2)](p_2p_1)^{-1}, \end{aligned} \quad (14)$$

$$\begin{aligned} v_{2ty} = &[-q_{1ty}q_2 + q_{2t}q_{1y} + q_{2y}q_{1t} - q_1q_{2ty} \\ &+ r(t)(q_{2y}q_1 - q_{1y}q_2)](q_1q_2)^{-1}, \end{aligned} \quad (15)$$

where $r(t)$ is an arbitrary function of t. Therefore, the solution (6) with the ansatz (11) exists, as anticipated, only if the additional constraints (12), (14) and (15) are imposed.

Next, we exchange the positions of q_1 and q_2 in the expression for g, while f is still the original expression (11), that is

$$\begin{aligned} f &= a_1'p_1(x,t)q_1(y,t) + a_2'p_2(x,t)q_2(y,t), \\ g &= b_1'p_1(x,t)q_2(y,t) + b_2'p_2(x,t)q_1(y,t). \end{aligned} \quad (16)$$

It is then fairly straightforward to show that (6) with (16) satisfies the bilinear forms (8) and (9) provided

$$\begin{aligned} v_{1tx} = &\big[-2a_2'p_{1tx}a_1'b_1'p_1 + 2p_{2tx}a_2'^2b_2'p_2 + 2b_2'p_{1x}b_1'^2p_{1t} \\ &- 2b_2'p_{2x}p_{2t}a_2'^2 + R(t)(p_1p_{2x} - p_{1x}p_2)\big] \\ &\cdot\big[a_2'(b_1'p_1^2a_1' - a_2'b_2'p_2^2)\big]^{-1}, \end{aligned} \quad (17)$$

$$v_{2ty} = \left[2q_{2ty}a_2'^2 b_1' q_2 - 2a_2' b_2' q_{1ty} a_1' q_1 + 2b_1' q_{1t} b_2'^2 q_{1y} - 2b_1' q_{2y} q_{2t} a_2'^2 + R(t)(q_{1y}q_2 - q_{2y}q_1)\right] \cdot \left[a_2'(b_2' q_1^2 a_1' - a_2' b_1' q_2^2)\right]^{-1}, \quad (18)$$

where $R(t)$ is an arbitrary function of t, a_i', b_i' $(i = 1,2)$ are constants and the constraint is

$$a_1' a_2' - b_1' b_2' = 0. \quad (19)$$

Finally, if we assume

$$f = c_0 + c_1 p(x,t) + c_2 q(y,t) + c_3 p(x,t) q(y,t),$$
$$g = d_0 + d_1 p(x,t) + d_2 q(y,t) + d_3 p(x,t) q(y,t), \quad (20)$$

where $c_i, d_i (i = 0..3)$ are constants. Then the solution (6) with (20) exists for

$$v_{1xx} = [s(t)(C_2 p^2 + C_1 p + C_0) - p_t - p_{xxx}](3p_x)^{-1}, \quad (21)$$

$$v_{2yy} = -[s(t)(B_2 q^2 + B_1 q + B_0) + q_t + q_{yyy}](3q_y)^{-1}, \quad (22)$$

where $s(t)$ is an arbitrary function of t and

$$\begin{aligned} C_0 &= d_3 c_0 d_2 - c_2^2 c_1 + c_2 c_3 c_0 - c_2 d_2 d_1, \\ C_1 &= -d_3 c_2 d_1 + d_3^2 c_0 - c_3 c_2 c_1 + c_3^2 c_0 \\ &\quad + d_3 d_2 c_1 - c_3 d_2 d_1, \\ C_2 &= d_3^2 c_1 - d_3 d_1 c_3, \\ B_0 &= c_1 c_3 c_0 + d_3 c_0 d_1 - c_1 d_2 d_1 - c_1^2 c_2, \\ B_1 &= c_3^2 c_0 + d_3 c_2 d_1 - d_3 d_2 c_1 - c_3 d_2 d_1 \\ &\quad + d_3^2 c_0 - c_3 c_2 c_1, \\ B_2 &= d_3^2 c_2 - d_3 d_2 c_3. \end{aligned} \quad (23)$$

The auxiliary condition requires that

$$d_3 d_0 - d_2 d_1 + c_3 c_0 - c_2 c_1 = 0. \quad (24)$$

Till now, we have found three different choices for the functions f and g in (6), namely first (11) with (12), (14) and (15), second (16) with (17) – (19), and third (20) with (21) – (24), respectively. Substituting them into (6), we can obtain three different types of variable separated solutions for the 2dim sine-Gordon system (4) and (5). Since some arbitrary, lower dimensional functions (like p), have been included in these forms, starting from any one of the three choices for the fields (or potential), many localized excitations such as solitoffs, multidromions, lumps, breathers, instantons, ring solitons, peakons, multivalued (folded) solitary waves (FSWs) are all admitted by the 2dim sine-Gordon system if we choose the arbitrary functions carefully to avoid the singularities. One also has to take proper care of the constraints when choosing the seed solutions and the constants a, b, etc.

3. Interaction Between some Typical Solitons

Actually, different choices of the arbitrary functions of one space variable only (like p) correspond to different choices of possible boundary conditions for the fields (or potentials). It is quite important but difficult to investigate the interaction properties for all the possible localized excitations. We therefore focus our attention on listing only two interactions between some typical localized excitations from the new types of variable separated solutions obtained in this paper for 2dim sine-Gordon system, which were not reported in the previous literature to the best of our knowledge.

Case (i) Interaction between two periodic solitons: If some periodic functions in the space variables are included in the functions (like p), we obtain some types of periodic excitations. To see the completely elastic interaction between two periodic solitons, we plot the time dependence. Choose e.g. f and g as

$$\begin{aligned} f &= 1 - \operatorname{sech}(3x - 16t) - \operatorname{sech}(4x) \\ &\quad + [-\operatorname{sech}(3x - 16t) - \operatorname{sech}(4x)]\cos(0.5y), \\ g &= 1 + [-\operatorname{sech}(3x - 16t) - \operatorname{sech}(4x)]\cos(0.5y), \end{aligned} \quad (25)$$

which corresponds to ansatz (20). The relevant functions and constants are then

$$\begin{aligned} &p = -\operatorname{sech}(3x - 16t) - \operatorname{sech}(4x),\ q = \cos(0.5y), \\ &c_0 = c_1 = c_3 = d_0 = d_3 = 1, \\ &c_2 = d_1 = d_2 = 0. \end{aligned} \quad (26)$$

In Fig. 1a – c, one can see that two periodic solitons for the field u in (6) with (25) pass through each other and completely preserve their shapes and velocities both for the head-on collision and the pursue collision. The expressions for two separated periodic solitons are as follows:

$$\frac{f}{g} = \frac{1 - \operatorname{sech}(3x - 16t) - \operatorname{sech}(3x - 16t)\cos(0.5y)}{1 - \operatorname{sech}(3x - 16t)\cos(0.5y)}, \quad (27)$$

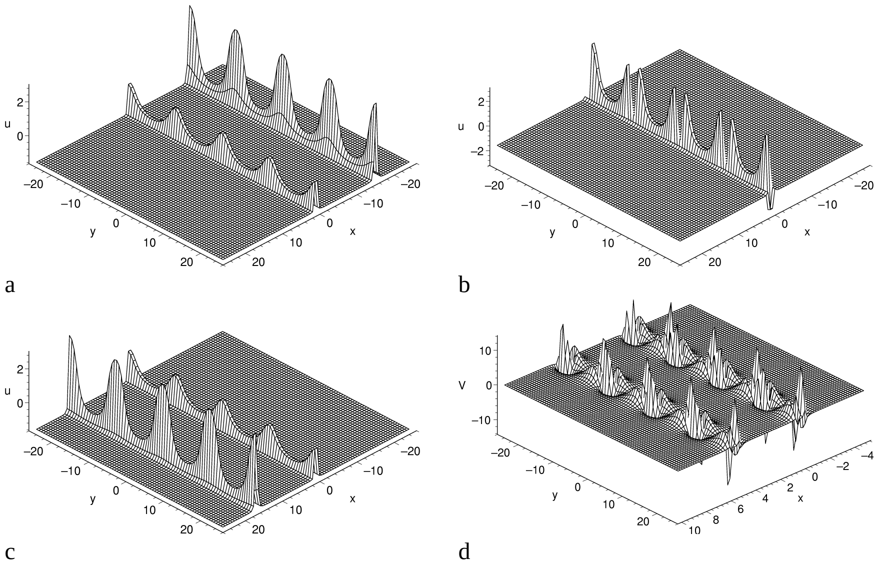

Fig. 1. (a) – (c) Evolution plot for the field u in (6) with the choices (25) at the times (a) $t = -3$, (b) $t = 0$, (c) $t = 3$, respectively; (d) The plot of the potential V is given by (29) at $t = 1$.

and

$$\frac{f}{g} = \frac{1 - \operatorname{sech}(4x) - \operatorname{sech}(4x)\cos(0.5y)}{1 - \operatorname{sech}(4x)\cos(0.5y)}. \qquad (28)$$

In Fig. 1b, two solitons form into a line, providing a temporary intermediate state, which propagates with the mean velocity of the two original solitons. After the formation of this state they begin to separate, restoring their shape. In addition to the completely elastic interaction, phase shifts are observed that follow the interaction. The velocity of one periodic soliton is fixed as zero, that is, it does not move with time. Clearly, before the interaction, the static periodic soliton is located at $x = -16$ while after the interaction, it shifts to $x = 16$. Fig. 1d is the plot of the potential function

$$V = (\ln(f^2 + g^2))_{xy}, \qquad (29)$$

where the functions g and f are determined by (25) with $t = 1$.

Case (ii) Interaction between two foldons: Multi-valued (folded) solitary waves (FSWs) are very useful localized excitations since there exist very complicated folded phenomena such as many kinds of folded biosystems in nature. The foldon is one kind of FSWs, whose interaction is completely elastic. In Fig. 2, the interaction between two foldons is plotted for u in (6), with the functions chosen as

$$\frac{f}{g} = \frac{2+3\operatorname{sech}(\zeta)+\operatorname{sech}(-\zeta+2t)+\operatorname{sech}(\theta)}{1+[1+3\operatorname{sech}(\zeta)+\operatorname{sech}(-\zeta+2t)](1+\operatorname{sech}(\theta))}, \qquad (30)$$

with

$$\begin{aligned} x &= \zeta - 3.5\tanh(\zeta)^2 + 2.5\tanh(-\zeta + 0.25t),\\ y &= \eta - 4.5\tanh(\theta) + 2\tanh(1.5\theta). \end{aligned} \qquad (31)$$

In (30) and (31) we understand that ζ and θ are multi-valued functions in certain regions of x and y respectively. So f/g is a multi-valued function of x and y in these regions though it is a single-valued function of ζ and θ. For simplicity, we also choose the velocity of the the big foldon to be zero. From Fig. 2 a – f, we can see that the “worm” (small one) is located at the position $x = 0.25t$, which varies with the time t. In Fig. 2 b the big foldon is slightly deformed and also its position shows a small shift, because it has already met the “worm”. The “worm” has made some effects on it which result in the change of the big foldon shown in Fig. 2 b at time $t = -11$. Its location has moved to a neighboring position. These changes reveal that the big foldon suffers a phase shift during the collision with the “worm”. Thereafter, the big foldon is stationarily located at this new position, again with a phase shift 5.

4. Summary and Discussion

In summary, because of the existence of arbitrary seed solution $\{v_1, v_2\}$, we obtain three new types of variable separated solution for the 2dim sine-Gordon system utilizing VSA. The arbitrary functions in these

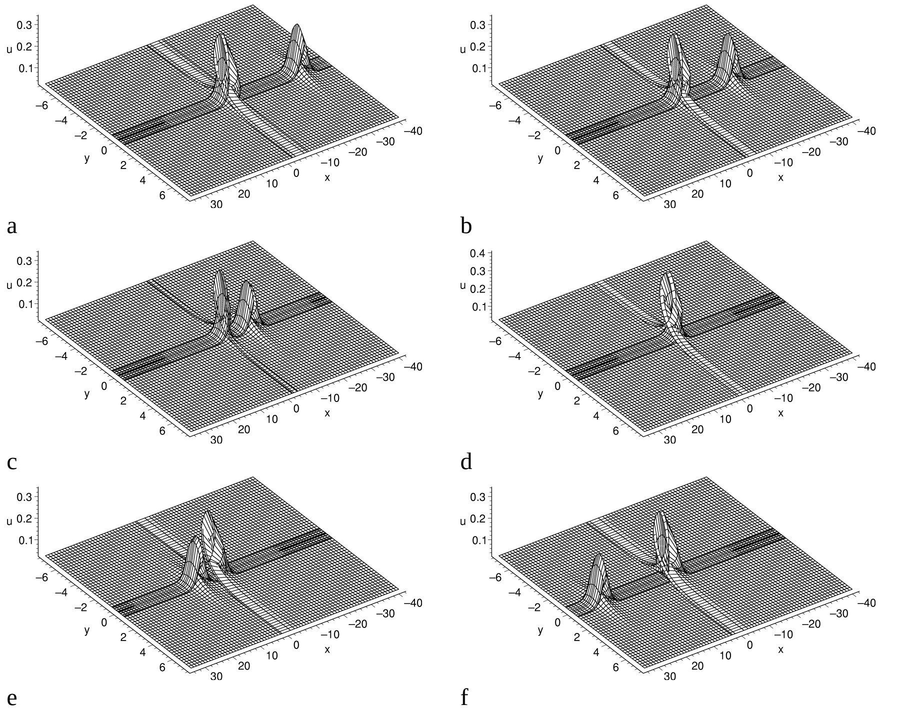

Fig. 2. Evolution plot for the field u in (6) with the choices (30) and (31) at the times (a) $t = -15$, (b) $t = -11$, (c) $t = -6$, (d) $t = 0$, (e) $t = 6$, (f) $t = 15$, respectively.

three types of ansatz can be properly harnessed to generate abundant types of excitations either localized or non-localized. It is quite important but difficult to investigate the interaction properties for all the possible localized excitations. In this paper, we have only discussed two examples of elastic interactions, one between two periodic solitons, another one between two foldons. Generally speaking, to follow the interaction course or to judge the properties of the interaction from the mathematical expressions is somewhat indirect and confusing compared with the clear and vivid pictures. So we have shown these two interactions graphically in a sequence of times. In this paper, there are only four arbitrary functions at most in the variable separated solutions, thus the question arises of whether we can introduce more characteristic functions into the variable separated solution. Actually, that is an avenue for our further study and investigation.

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